An introduction to interpolation between random matrices and free operators

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The aim of this document is to provide an elementary introduction to the strategy that we developed in [2, 5], notably by making use of a new idea which considerably shortens the original proof. It provides a very short proof of the main result of [3] which proves the convergence of the norm of any polynomials in independent GUE matrices. Besides I made it a point to make it as accessible as possible for people unfamiliar with free probability. Thus the first three sections provide a mostly self-contained introduction to the topic which is sufficient to understand the proof in the last section. If you are already familiar with free probability you can probably skip most of it, however I would recommend to check the construction of $\mathcal{A}_N$ which leads to Theorem 1.4 since it is instrumental in the proof of Theorem 4.1.

1 Usual definitions in free probability

Let us begin by recalling the following definitions from free probability. Although some of them might be a bit abstract for our need, we will try to specify exactly what we need afterward.

Definition 1.1.

- *A $\mathcal{C}^*$-probability space $(\mathcal{A}, *, \tau, \|.\|)$ is a unital $\mathcal{C}^*$-algebra $(\mathcal{A}, *, \|.\|)$ endowed with a state τ, i.e. a linear map $\tau : \mathcal{A} \to \mathbb{C}$ satisfying $\tau(1_{\mathcal{A}}) = 1$ and $\tau(a^*a) \geq 0$ for all $a \in \mathcal{A}$. In this paper we always assume that τ is a trace, i.e. that it satisfies $\tau(ab) = \tau(ba)$ for any $a, b \in \mathcal{A}$. An element of $\mathcal{A}$ is called a (noncommutative) random variable. We will always work with a faithful trace, namely, for $a \in \mathcal{A}$, $\tau(a^*a) = 0$ if and only if $a = 0$.*

- *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be $*$-subalgebras of $\mathcal{A}$, having the same unit as $\mathcal{A}$. They are said to be free if for all k, for all $a_i \in \mathcal{A}_{j_i}$ such that $j_1 \neq j_2$, $j_2 \neq j_3$, $\dots$, $j_{k-1} \neq j_k$:*

$$\tau\Big((a_1 - \tau(a_1))(a_2 - \tau(a_2)) \dots (a_k - \tau(a_k))\Big) = 0. \tag{1.1}$$

  Families of noncommutative random variables are said to be free if the $*$-subalgebras they generate are free. Note in particular that if X and Y are free, then

$$\tau(XY) = \tau(X)\tau(Y) \tag{1.2}$$

- *A family of noncommutative random variables $x = (x_1, \dots, x_d)$ is called a free semicircular system when the noncommutative random variables are free, self-adjoint ($x_i = x_i^*$), and for all k in $\mathbb{N}$ and i, one has*

$$\tau(x_i^k) = \begin{cases} c_{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{else.} \end{cases}$$

  where c_n is the n-th Catalan number.

Although we will introduce an alternative to this theorem. It is important to note that thanks to [4, Theorem 7.9], that we recall below, one can consider free copies of any noncommutative random variable.

Theorem 1.2. *Let $(\mathcal{A}_i, \phi_i)_{i \in I}$ be a family of $\mathcal{C}^*$-probability spaces such that the functionals $\phi_i : \mathcal{A}_i \to \mathbb{C}$, $i \in I$, are faithful traces. Then there exist a $\mathcal{C}^*$-probability space $(\mathcal{A}, \phi)$ with ϕ a faithful trace, and a family of norm-preserving unital $*$-homomorphism $W_i : \mathcal{A}_i \to \mathcal{A}$, $i \in I$, such that:*

- $\phi \circ W_i = \phi_i$, $\forall i \in I$.
- *The unital $\mathcal{C}^*$-subalgebras form a free family in* $(\mathcal{A}, \phi)$.

The main reason for us to introduce this theorem is because in this document we want to consider the free product of $M_N(\mathbb{C})$ with the algebra generated by free semi-circular variables. However we will do so with a specific construction after fixing a few notations concerning the spaces and traces that we use in this paper.

Definition 1.3.
- $(e_u)_{1\leq N}$ *is the canonical basis of* $\mathbb{C}^N$.
- $\mathrm{Tr}_N : A \mapsto \sum_{1\leq u\leq N} e_u^* A e_u$ *is the non-renormalized trace on* $\mathbb{M}_N(\mathbb{C})$.
- *We denote* $E_{r,s} = e_r^* e_s$ *the matrix with coefficients equal to* 0 *except in* (r, s) *where it is equal to one.*

In order to interpolate our random matrices with free operators, we now need to define a space in which they can both live in simultaneously. To do so, we define $(\widetilde{\mathcal{A}}_N, \widetilde{\tau}_N)$ as the free product of $\mathbb{M}_N(\mathbb{C})$ with a system of d free semicircular variable, that is the $\mathcal{C}^*$-probability space built in Theorem 1.2. Note that when restricted to $\mathbb{M}_N(\mathbb{C})$, $\widetilde{\tau}_N$ is just the regular renormalized trace on matrices. However this definition is not especially intuitive, besides we will need a different construction in the rest of the document.

We fix $d, N \in \mathbb{N}$, thanks to the help of the so-called full Fock space, one can easily build an explicit $\mathcal{C}^*$-probability spaces $(\mathcal{A}, *, \tau, \|.\|)$ where τ is a faithfull trace and in which there exists a free semicircular system $(x^i_{r,s})_{1\leq i\leq d, 1\leq r\leq s\leq N} \cup (y^i_{r,s})_{1\leq i\leq d, 1\leq r<s\leq N}$. For a proof we refer to Corollary 7.17 of [4]. Next we fix $\mathcal{A}_N = \mathbb{M}_N(\mathcal{A})$, thus if $\mathbf{1}$ is the unit of $\mathcal{A}$, one can easily view $\mathbb{M}_N(\mathbb{C})$ as a subalgebra of $\mathcal{A}_N$ thanks to the morphism $(a_{r,s}) \in \mathbb{M}_N(\mathbb{C}) \mapsto (a_{r,s}\mathbf{1}) \in \mathcal{A}_N$. We also define $x^N_i \in \mathcal{A}_N$ with

$$\sqrt{N}\ (x^N_i)_{r,s} = \begin{cases} \frac{x^i_{r,s} + \mathbf{i}\ y^i_{r,s}}{\sqrt{2}} & \text{if } r < s, \\ x^i_{r,s} & \text{if } r = s, \\ \frac{x^i_{s,r} - \mathbf{i}\ y^i_{s,r}}{\sqrt{2}} & \text{if } r > s. \end{cases}$$

We endow $\mathcal{A}_N$ with the involution ${(a_{i,j})_{1\leq i,j\leq N}}^* = (a^*_{j,i})_{1\leq i,j\leq N}$ and the trace $\tau_N : A \in \mathcal{A}_N \mapsto \tau\left(\frac{1}{N}\mathrm{Tr}_N(A)\right)$. Then one has the following theorem.

Theorem 1.4. *With the trace and the involution defined as above, $\mathcal{A}_N$ is a $\mathcal{C}^*$-probability spaces. Besides the family $(x^N_i)_{1\leq i\leq d}$ is a free semicircular system, and it is free from $\mathbb{M}_N(\mathbb{C})$.*

We delay the proof of this theorem to the end of the next section. Let us finally note that in the last section we will drop the N and simply write x_i instead x^N_i since thanks to the previous theorem, we have that the trace of a polynomial in x^N is the same for any N (i.e. the trace of a polynomial evaluated in a free semi-circular system).

2 Non-commutative polynomials and derivatives

Let $\mathcal{A}_{d,2r} = \mathbb{C}\langle X_1, \dots, X_d, Y_1, \dots, Y_{2r}\rangle$ be the set of noncommutative polynomial in $d + 2r$ variables. We set $q = 2r$ to simplify notations. Let us define several maps which we use frequently in the sequel. First, for $A, B, C \in \mathcal{A}_{d,q}$, let

$$A \otimes B \# C = ACB,\ m(A \otimes B) = BA. \tag{2.1}$$

We define an involution $*$ on $\mathcal{A}_{d,q}$ by $X^*_i = X_i$, $Y^*_i = Y_{i+r}$ if $i \leq d + r$, $Y^*_i = Y_{i-r}$ else. Then we extend it to $\mathcal{A}_{d,q}$ by linearity and the formula $(\alpha PQ)^* = \overline{\alpha} Q^* P^*$. $P \in \mathcal{A}_{d,q}$ is said to be self-adjoint if $P^* = P$. Self-adjoint polynomials have the property that if $x_1, \dots, x_d, z_1, \dots, z_r$ are elements of a $\mathcal{C}^*$-algebra such that $x_1, \dots, x_d$ are self-adjoint, then so is $P(x_1, \dots, x_d, z_1, \dots, z_r, z^*_1, \dots, z^*_r)$. This leads us to define the following space.

Definition 2.1. *We define $\mathcal{F}_{d,q}$ to be the $*$-algebra generated by $\mathcal{A}_{d,q}$ and the family*

$$\left\{(z-P)^{-1} \mid P \in \mathcal{A}_{d,q} \text{ is self-adjoint and } z \in \mathbb{C} \setminus \mathbb{R}\right\}.$$

We then define the following notion of non-commutative differential on this space.

Definition 2.2. *If $1 \leq i \leq d$, we define the noncommutative derivative $\partial_i : \mathcal{F}_{d,q} \longrightarrow \mathcal{F}_{d,q} \otimes \mathcal{F}_{d,q}$ by induction with the following formula,*

$$\begin{gathered} \forall S,T \in \mathcal{F}_{d,q}, \quad \partial_i(ST) = \partial_i S\,(1 \otimes T) + (S \otimes 1)\,\partial_i T, \\ \forall i,j, \quad \partial_i X_j = \delta_{i,j} 1 \otimes 1. \\ \forall i,j, \quad \partial_i Y_j = 0 \otimes 0. \\ \forall P \in \mathcal{A}_{d,q} \text{ self-adjoint}, \quad \partial_i (z-P)^{-1} = \left((z-P)^{-1} \otimes 1\right) \partial_i P \left(1 \otimes (z-P)^{-1}\right). \end{gathered} \tag{2.2}$$

Similarly, with m as in (2.1)*, one defines the cyclic derivative $D_i : \mathcal{F}_{d,q} \longrightarrow \mathcal{F}_{d,q}$ for $S \in \mathcal{F}_{d,q}$ by*

$$D_i S = m \circ \partial_i S\ .$$

In order to better visualize this notion of non-commutative differentials, it is worth noting that if $S \in \mathcal{F}_{d,q}$, given r matrices $(Z_1, \dots, Z_r)$ and $\mathbb{M}_N(\mathbb{C})_{sa}$ the set of Hermitian matrices of size N, one can define the following map,

$$\begin{array}{cccc} L^S & : & (\mathbb{M}_N(\mathbb{C})_{sa})^d & \to & \mathbb{M}_N(\mathbb{C}) \\ & & X & \mapsto & S(X,Z,Z^*), \end{array}$$

and by induction it is easy to see that the differential of this map in X is the following,

$$d(L^S)_X : H \in (\mathbb{M}_N(\mathbb{C})_{sa})^d \mapsto \sum_i \partial_i S(X,Z,Z^*) \# H_i.$$

Besides the map ∂_i is related to the so-called Schwinger-Dyson equations on semicircular variable thanks to the following proposition.

Proposition 2.3. *Let $x = (x_1, \dots, x_d)$ be a free semicircular system, $z = (z_1, \dots, z_r)$ be noncommutative random variables free from x, if the family (x,z) belongs to the $\mathcal{C}^*$-probability space $(\mathcal{A}, *, \tau, \|.\|)$, then for any $Q \in \mathcal{F}_{d,q}$,*

$$\tau(x_i\ Q(x,z,z^*)) = \tau \otimes \tau(\partial_i Q(x,z,z^*)). \tag{2.3}$$

Proof. If Q is a polynomial in x_i, by linearity one can assume that Q is a monomial, but then we can rewrite Equation (2.3) as

$$\tau(x_i^{k+1}) = \sum_{l=0}^{k-1} \tau(x_i^l)\tau(x_i^{k-1-l}),$$

which is the induction formula satisfied by the Catalan numbers, i.e. by the moment of a semicircular variable.

If we now assume that $Q \in \mathcal{A}_{d,q}$, by induction we can write Q as a linear combination of product of polynomials $P_1 \dots P_k$ where each P_j is a polynomial in (z,z^*) or in a single x_l. Besides we can assume that for any j, $\tau(P_j(x,z,z^*)) = 0$ and that we do not have two polynomials in the same variables following each other. Then if $k = 1$, we are either in the previous situation or P_1 is a polynomial in (z,z^*), hence Equation (2.3) is satisfied in both case. Thus let us assume that $k \geq 2$, if P_1 is not a polynomial in x_i, thanks to Equation (1.1), we have that

$$\tau(x_i\ Q(x,z,z^*)) = 0.$$

If P_1 is a polynomial in x_i, since $k \geq 2$, we also have that

$$\begin{aligned} \tau(x_i\ Q(x,z,z^*)) &= \tau\left(\Big(x_i P_1(x,z,z^*) - \tau(x_i P_1(x,z,z^*))\Big) P_2(x,z,z^*) \dots P_k(x,z,z^*)\right) \\ &\quad + \tau(x_i P_1(x,z,z^*))\tau\left(P_2(x,z,z^*) \dots P_k(x,z,z^*)\right) \\ &= 0. \end{aligned}$$

Similarly, since we assumed that $k \geq 2$, we also have that

$$\tau \otimes \tau(\partial_i Q(x, z, z^*)) = 0.$$

Consequently, Equation (2.3) is satisfied for any polynomial. If $Q \in \mathcal{F}_{d,q}$, then given $P \in \mathcal{A}_{d,q}$ self-adjoint, for $z \in \mathbb{C}$ such that $|z| > \|P(x, z, z^*)\|$, one can expand $(z - P(x, z, z^*))^{-1}$ as a power series in z^{-1}, use the first part of the proof for polynomial and the equality

$$\sum_{j \geq 0} \partial_i P^j z^{-j-1} = \left((z-P)^{-1} \otimes 1\right) \partial_i P \left(1 \otimes (z-P)^{-1}\right).$$

Finally we conclude for any $z \in \mathbb{C} \setminus \mathbb{R}$ by analyticity.

□

Proof of Theorem 1.4. Let us fix $Z_1, \dots, Z_q \in \mathbb{M}_N(\mathbb{C})$, then for any polynomial $P \in \mathcal{A}_{d,q}$, thanks to Proposition 2.3, one can show that for any i,

$$\tau_N(x_i^N \; P(x^N, Z)) = \tau_N \otimes \tau_N(\partial_i P(x^N, Z)).$$

Hence by induction on the degree of P, we get that

$$\tau_N(P(x^N, Z)) = \widetilde{\tau}_N(P(x, Z)),$$

where x is a free semicircular system, free from $M_N(\mathbb{C})$, which we view as element of $\widetilde{\mathcal{A}}_N$. Hence the conclusion. □

3 GUE random matrices

In this section we define Gaussian random matrices and states a few useful properties about them.

Definition 3.1. *A GUE random matrix X^N of size N is a self-adjoint matrix whose coefficients are random variables with the following laws:*

- *For $1 \leq i \leq N$, the random variables $\sqrt{N} X_{i,i}^N$ are independent centered Gaussian random variables of variance 1.*
- *For $1 \leq i < j \leq N$, the random variables $\sqrt{2N}\ \Re X_{i,j}^N$ and $\sqrt{2N}\ \Im X_{i,j}^N$ are independent centered Gaussian random variables of variance 1, independent of $\left(X_{i,i}^N\right)_i$.*

When doing computations with Gaussian variables, the main tool that we use is Gaussian integration by parts. It can be summarized into the following formula, if Z is a centered Gaussian variable with variance one and $f \in \mathcal{C}^1(\mathbb{R})$, then by integration by parts,

$$\mathbb{E}[Zf(Z)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\ x e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f'(x) e^{-x^2/2} dx = \mathbb{E}[f'(Z)]. \tag{3.1}$$

A direct consequence of this, is that if x and y are centered Gaussian variables with variance one, and $Z = \frac{x + \mathbf{i}y}{\sqrt{2}}$, then with $f \in \mathcal{C}^1(\mathbb{C})$,

$$\mathbb{E}[Zf(Z, \bar{Z})] = \mathbb{E}[\partial_1 f(Z, \bar{Z})] \quad \text{and} \quad \mathbb{E}[\bar{Z} f(Z, \bar{Z})] = \mathbb{E}[\partial_2 f(Z, \bar{Z})]. \tag{3.2}$$

For example we have that given a GUE random matrix X^N, one can write $X^N = \frac{1}{\sqrt{N}}(x_{r,s})_{1 \leq r,s \leq N}$ and then for any polynomial Q,

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}_N\left(X^N\ Q(X^N)\right)\right] &= \frac{1}{N^{3/2}}\sum_{r,s}\mathbb{E}\left[x_{r,s}\ \ \operatorname{Tr}_N\left(E_{r,s}\ Q(X^N)\right)\right] \\
&= \frac{1}{N^{3/2}}\sum_{r,s}\mathbb{E}\left[\operatorname{Tr}_N\left(E_{r,s}\ \partial_{x_{r,s}}Q(X^N)\right)\right] \\
&= \frac{1}{N^{2}}\sum_{r,s}\mathbb{E}\left[\operatorname{Tr}_N\left(E_{r,s}\ \partial Q(X^N)\# E_{s,r}\right)\right] \\
&= \frac{1}{N^{2}}\sum_{r,s}\mathbb{E}\left[e_s^*\left(\partial Q(X^N)\# e_s e_r^*\right)e_r\right] \\
&= \mathbb{E}\left[\left(\frac{1}{N}\operatorname{Tr}_N\right)^{\otimes 2}\left(\partial Q(X^N)\right)\right].
\end{aligned}$$

4 Main theorem

In this section we prove the main result of this document. It is a slightly weaker version of Lemma 3.2 of [5]. Indeed, in order to obtain a full asymptotic expansion, one needs to be able to bootstrap, i.e. to reapply this lemma to the term under the integral in Equation (4.1). Although intuitively the proof of the full lemma is rather similar, it does necessitate to introduce much heavier notations so we will avoid doing so. Note finally that in this proof we introduce a new strategy which allows us to avoid having to approximate our free variables by random matrices, which makes the proof considerably shorter.

Theorem 4.1. *Let the following objects be given,*

- $X^N = (X_1^N, \dots, X_d^N)$ *independent GUE matrices of size* N,
- x, z^1, z^2 *free families of* d *free semi-circular variables, defined as in Theorem 1.4,*
- $Z^N = (Z_1^N, \dots, Z_q^N)$ *deterministic matrices and their adjoints,*
- $z_{s,t}^1 = \Big((1-e^{-s})^{1/2} z^1 + (e^{-s}-e^{-t})^{1/2} x + e^{-t/2} X^N, Z^N\Big)$,
- $z_{s,t}^2$ *defined similarly but with* z^2 *instead of* z^1,
- $\widetilde{z}_{s,t}^1, \widetilde{z}_{s,t}^2$ *defined similarly but where we replaced* z^1, z^2, x *by free copies,*
- $Q \in \mathcal{F}_{d,q}$.

Then, for any N,

$$\begin{aligned}
&\mathbb{E}\left[\tau_N\Big(Q\left(X^N, Z^N\right)\Big)\right] - \tau_N\Big(Q\left(x, Z^N\right)\Big) \\
&= \frac{1}{2N^2}\sum_{1\le i,j\le d}\int_0^\infty\int_0^t e^{-s-t}\ \mathbb{E}\Big[\tau_N\Big(\Theta^{z_{s,t}^1, \widetilde{z}_{s,t}^1, \widetilde{z}_{s,t}^2, z_{s,t}^2}\circ(\partial_j\otimes\partial_j)\circ\partial_i D_i Q\Big)\Big] ds\ dt.
\end{aligned} \tag{4.1}$$

where for any $A, B, C, D \in \mathcal{F}_{d,q}$,

$$\Theta^{z_{s,t}^1, \widetilde{z}_{s,t}^1, \widetilde{z}_{s,t}^2, z_{s,t}^2}(A\otimes B\otimes C\otimes D) = B(z_{s,t}^1)A(\widetilde{z}_{s,t}^1)D(\widetilde{z}_{s,t}^2)C(z_{s,t}^2).$$

Proof. With

$$z_t = \left((1-e^{-t})^{1/2}x + e^{-t/2}X^N, Z^N\right),$$

we have,

$$\mathbb{E}\left[\tau_N\Big(Q\left(X^N,Z^N\right)\Big)\right]-\tau_N\Big(Q\left(x,Z^N\right)\Big)=-\int_0^\infty \mathbb{E}\left[\frac{d}{dt}\tau_N\Big(Q\left(z_t\right)\Big)\right]dt.$$

We can compute

$$\frac{d}{dt}\tau_N\Big(Q\left(z_t\right)\Big)=\frac{e^{-t}}{2}\sum_{1\le i\le d}\tau_N\left(D_iQ\left(z_t\right)\left(\frac{x_i}{(1-e^{-t})^{1/2}}-e^{t/2}X_i^N\right)\right).$$

Thus thanks to the Schwinger-Dyson equations (see Proposition 2.3) and Gaussian integration by parts (see (3.2)), we get that

$$\mathbb{E}\left[\frac{d}{dt}\tau_N\Big(Q\left(z_t\right)\Big)\right]=\mathbb{E}\left[\frac{e^{-t}}{2}\sum_{1\le i\le d}\left(\tau_N\otimes\tau_N\Big(\partial_iD_iQ\left(z_t\right)\Big)-\frac{1}{N}\sum_{1\le u,v\le N}\tau_N\Big(E_{u,v}\ \partial_iD_iQ\left(z_t\right)\#E_{v,u}\Big)\right)\right]. \tag{4.2}$$

For $A,B\in\mathcal{F}_{d,q}$, let

$$\Lambda_{N,t}:=\tau_N\Big(A\left(z_t\right)\Big)\tau_N\Big(B\left(z_t\right)\Big)-\frac{1}{N}\sum_{1\le u,v\le N}\tau_N\Big(E_{u,v}A\left(z_t\right)E_{v,u}B\left(z_t\right)\Big).$$

Remark 4.2. *Interestingly from there on we do not need to assume that we are working with GUE random matrices anymore. In the definition of $\Lambda_{N,t}$ one could very well assume that X^N are deterministic matrices. In particular, if X^N are Wigner matrices or GOE random matrices then most of the proof is the same, except that we will have to deal with the extra term which will appear in (4.2) with the cumulant expansion.*

We now want to compute $\Lambda_{N,t}$. To do so, we use that $z_t=z^1_{0,t}=z^1_{0,t}$ as well as Equation (1.2) combined with the fact that z_t has the same distribution as $z^1_{t,t}$ or $z^2_{t,t}$ (i.e. the trace of any polynomial in z_t is the same as the trace of this polynomial evaluated in $z^1_{t,t}$ or $z^2_{t,t}$). This yields,

$$\begin{aligned}
\Lambda_{N,t}&=\tau_N\Big(A\left(z^1_{t,t}\right)\Big)\tau_N\Big(B\left(z^2_{t,t}\right)\Big)-\frac{1}{N}\sum_{u,v}\tau_N\Big(E_{u,v}A\left(z^1_{0,t}\right)E_{v,u}B\left(z^2_{0,t}\right)\Big)\\
&=\frac{1}{N^2}\sum_{u,v}\tau\Big(e_v^*A\left(z^1_{t,t}\right)e_v\Big)\tau\Big(e_u^*B\left(z^2_{t,t}\right)e_u\Big)-\frac{1}{N}\sum_{u,v}\tau_N\Big(E_{u,v}A\left(z^1_{0,t}\right)E_{v,u}B\left(z^2_{0,t}\right)\Big)\\
&=\frac{1}{N^2}\sum_{u,v}\tau\Big(e_v^*A\left(z^1_{t,t}\right)e_ve_u^*B\left(z^2_{t,t}\right)e_u\Big)-\frac{1}{N}\sum_{u,v}\tau_N\Big(E_{u,v}A\left(z^1_{0,t}\right)E_{v,u}B\left(z^2_{0,t}\right)\Big)\\
&=\frac{1}{N}\sum_{u,v}\tau_N\Big(E_{u,v}A\left(z^1_{t,t}\right)E_{v,u}B\left(z^2_{t,t}\right)\Big)-\tau_N\Big(E_{u,v}A\left(z^1_{0,t}\right)E_{v,u}B\left(z^2_{0,t}\right)\Big)\\
&=\frac{1}{N}\sum_{u,v}\int_0^t\frac{d}{ds}\tau_N\Big(E_{u,v}A\left(z^1_{s,t}\right)E_{v,u}B\left(z^2_{s,t}\right)\Big)\,ds.
\end{aligned}$$

Besides,

$$\begin{aligned}
&\frac{d}{ds}\tau_N\Big(E_{u,v}A\left(z^1_{s,t}\right)E_{v,u}B\left(z^2_{s,t}\right)\Big)\\
&=\sum_j\frac{e^{-s}}{2}\left(\tau_N\left(E_{u,v}\ \partial_jA\left(z^1_{s,t}\right)\#\left(\frac{z^1}{(1-e^{-s})^{1/2}}-\frac{x}{(e^{-s}-e^{-t})^{1/2}}\right)\ E_{v,u}B\left(z^2_{s,t}\right)\right)\right.\\
&\qquad\left.+\tau_N\left(E_{u,v}A\left(z^1_{s,t}\right)E_{v,u}\ \partial_jB\left(z^2_{s,t}\right)\#\left(\frac{z^2}{(1-e^{-s})^{1/2}}-\frac{x}{(e^{-s}-e^{-t})^{1/2}}\right)\right)\right).
\end{aligned}$$

Thus if we write $\partial_j A = \sum_{A=A_1X_jA_2} A_1\otimes A_2$ and $\partial_j B = \sum_{B=B_1X_jB_2} B_1\otimes B_2$, then thanks to Proposition 2.3,

$$\begin{aligned}
&\tau_N\left(E_{u,v}\ \partial_j A\left(z^1_{s,t}\right)\#\left(\frac{z^1}{(1-e^{-s})^{1/2}}-\frac{x}{(e^{-s}-e^{-t})^{1/2}}\right)\ E_{v,u}B\left(z^2_{s,t}\right)\right)\\
&=\sum_{A=A_1X_jA_2}\tau_N\left(\left(\frac{z^1}{(1-e^{-s})^{1/2}}-\frac{x}{(e^{-s}-e^{-t})^{1/2}}\right)A_2\left(z^1_{s,t}\right)E_{v,u}B\left(z^2_{s,t}\right)E_{u,v}A_1\left(z^1_{s,t}\right)\right)\\
&=-\sum_{A=A_1X_jA_2,\ B=B_1X_jB_2}\tau_N\left(A_2\left(z^1_{s,t}\right)E_{v,u}B_1\left(z^2_{s,t}\right)\right)\ \tau_N\left(B_2\left(z^2_{s,t}\right)E_{u,v}A_1\left(z^1_{s,t}\right)\right)\\
&=-\frac{1}{N^2}\sum_{A=A_1X_jA_2,\ B=B_1X_jB_2}\tau\left(e_u^*B_1\left(z^2_{s,t}\right)A_2\left(z^1_{s,t}\right)e_v\right)\ \tau\left(e_v^*A_1\left(z^1_{s,t}\right)B_2\left(z^2_{s,t}\right)e_u\right)\\
&=-\frac{1}{N^2}\sum_{A=A_1X_jA_2,\ B=B_1X_jB_2}\tau\left(e_u^*B_1\left(z^2_{s,t}\right)A_2\left(z^1_{s,t}\right)E_{v,v}\ A_1\left(\tilde{z}^1_{s,t}\right)B_2\left(\tilde{z}^2_{s,t}\right)e_u\right).
\end{aligned}$$

Note that we used Equation (1.2) in the last line. Similarly, we also have

$$\begin{aligned}
&\tau_N\left(E_{u,v}A\left(z^1_{s,t}\right)E_{v,u}\ \partial_j B\left(z^2_{s,t}\right)\#\left(\frac{z^2}{(1-e^{-s})^{1/2}}-\frac{x}{(e^{-s}-e^{-t})^{1/2}}\right)\right)\\
&=-\frac{1}{N^2}\sum_{A=A_1X_jA_2,\ B=B_1X_jB_2}\tau\left(e_u^*B_1\left(z^2_{s,t}\right)A_2\left(z^1_{s,t}\right)E_{v,v}\ A_1\left(\tilde{z}^1_{s,t}\right)B_2\left(\tilde{z}^2_{s,t}\right)e_u\right).
\end{aligned}$$

Consequently we have that,

$$\begin{aligned}
\Lambda_{N,t}&=-\frac{1}{N^3}\sum_{u,v}\sum_j\sum_{A=A_1X_jA_2,\ B=B_1X_jB_2}\int_0^t e^{-s}\ \tau\left(e_u^*B_1\left(z^2_{s,t}\right)A_2\left(z^1_{s,t}\right)E_{v,v}\ A_1\left(\tilde{z}^1_{s,t}\right)B_2\left(\tilde{z}^2_{s,t}\right)e_u\right)ds\\
&=-\frac{1}{N^2}\sum_j\sum_{A=A_1X_jA_2,\ B=B_1X_jB_2}\int_0^t e^{-s}\ \tau_N\left(A_2\left(z^1_{s,t}\right)A_1\left(\tilde{z}^1_{s,t}\right)B_2\left(\tilde{z}^2_{s,t}\right)B_1\left(z^2_{s,t}\right)\right)ds\\
&=-\frac{1}{N^2}\sum_j\int_0^t e^{-s}\ \tau_N\left(\Theta^{z^1_{s,t},\tilde{z}^1_{s,t},\tilde{z}^2_{s,t},z^2_{s,t}}\left(\partial_j A\otimes\partial_j B\right)\right)ds.
\end{aligned}$$

Hence in combination with Equation (4.2),

$$\begin{aligned}
&\mathbb{E}\left[\tau_N\Big(Q\left(X^N,Z^N\right)\Big)\right]-\tau_N\Big(Q\left(x,Z^N\right)\Big)\\
&=\frac{1}{2N^2}\sum_{1\le i,j\le d}\int_0^\infty\int_0^t e^{-s-t}\ \mathbb{E}\Big[\tau_N\Big(\Theta^{z^1_{s,t},\tilde{z}^1_{s,t},\tilde{z}^2_{s,t},z^2_{s,t}}\circ(\partial_j\otimes\partial_j)\circ\partial_i D_i Q\Big)\Big]ds\ dt.
\end{aligned}$$

□

In particular, one has for any self-adjoint polynomial P that,

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}_N\left(\Big(z-P(X^N)\Big)^{-1}\right)\right]=\tau\left(\Big(z-P(x)\Big)^{-1}\right)+\mathcal{O}\left(\frac{1}{N^2|\Im(z)|^5}\right),$$

from whom one can easily deduce the strong convergence of a family of independent GUE random matrices (see the proof of Theorem 5.5.1 of [1], its main difficulty was the proof of Lemma 5.5.4 which consists in proving the equation above).

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